

RELATIONS BETWEEN SOME CLASSES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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ABSTRACT. We prove inclusion relations between generalized Waterman's and generalized Wiener's classes for functions of two variable.

The notion of function of bounded variation was introduced by C. Jordan [17]. Generalized this notion N. Wiener [30] has considered the class BV_p of functions. L. Young [31] introduced the notion of functions of Φ -variation. In [26] D. Waterman has introduced the following concept of generalized bounded variation.

Definition 1. Let $\Lambda = \{\lambda_n : n \geq 1\}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} (1/\lambda_n) = \infty$. A function f is said to be of Λ -bounded variation ($f \in \Lambda BV$), if for every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$ we have

$$\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty,$$

where $I_n = [a_n, b_n] \subset [0, 1]$ and $f(I_n) = f(b_n) - f(a_n)$.

If $f \in \Lambda BV$, then Λ -variation of f is defined to be the supremum of such sums, denoted by $V_{\Lambda}(f)$.

Properties of functions of the class ΛBV as well as the convergence and summability properties of their Fourier series have investigated in [22]-[29].

For everywhere bounded 1-periodic functions, Z. Chanturia [6] has introduced the concept of the modulus of variation.

H. Kita and K. Yoneda [18] studied generalized Wiener classes $BV(p(n) \uparrow p)$. They introduced

Definition 2. Let f be a finite 1-periodic function defined on the interval $(-\infty, +\infty)$. Δ is said to be a partition with period 1 if there is a set of points t_i for which

$$(1) \quad \cdots t_{-1} < t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} < \cdots,$$

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and $t_{k+m} = t_k + 1$ when $k = 0, \pm 1, \pm 2, \dots$, where m is any natural number. Let $p(n)$ be an increasing sequence such that $1 \leq p(n) \leq p$, $n \rightarrow \infty$, where $1 \leq p \leq +\infty$. We say that a function f belongs to the class $BV(p(n) \uparrow p)$ if

$$V(f, p(n) \uparrow p) \equiv \sup_{n \geq 1} \sup_{\Delta} \left\{ \left(\sum_{k=1}^m |f(I_k)|^{p(n)} \right)^{1/p(n)} : \inf_k |I_k| \geq \frac{1}{2^n} \right\} < +\infty.$$

We note that if $p(n) = p$ for each natural number, where $1 \leq p < +\infty$, then the class $BV(p(n) \uparrow p)$ coincides with the Wiener class V_p .

Properties of functions of the class $BV(p(n) \uparrow p)$ as well as the uniform convergence and divergence at point of their Fourier series with respect to trigonometric and Walsh system have been investigated in [9],[12],[19].

Generalizing the class $BV(p(n) \uparrow p)$ T. Akhobadze (see [1],[2]) has considered the $BV(p(n) \uparrow p, \varphi)$ and $B\Lambda(p(n) \uparrow p, \varphi)$ classes of functions.

The relation between different classes of generalized bounded variation was taken into account in the works of M. Avdispahic [4], A. Kovocik [20], A. Belov (see [5], Z. Chanturia [7]), T. Akhobadze [3] and M. Medvedieva [21], Goginava [11, 13].

Let f be a real and measurable function of two variable of period 1 with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points x, y from $I := [0, 1]$ we denote

$$f(I, y) := f(b, y) - f(a, y), \quad f(x, J) := f(x, d) - f(x, c)$$

and for the rectangle $A = (a, b) \times (c, d)$, we set

$$f(A) = f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E .

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^\infty$ we denote

$$\Lambda V_1(f) = \sup_{y \in T} \sup_{\{I_i\} \in \Omega} \sum_i \frac{|f(I_i, y)|}{\lambda_i},$$

$$\Lambda V_2(f) = \sup_{x \in T} \sup_{\{J_j\} \in \Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j}.$$

$$\Lambda V_{1,2}(f) = \sup_{\{I_i\}, \{J_j\} \in \Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

Definition 3. We say that the function f has Bounded Λ -variation on $I^2 := [0, 1] \times [0, 1]$ and write $f \in \Lambda BV$, if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function f has Bounded Partial Λ -variation and write $f \in P\Lambda BV$ if

$$PAV(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \dots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \rightarrow \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus, in what follows we suppose that

$$(2) \quad 1 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

In the case when $\lambda_n = n$, $n = 1, 2, \dots$ we say *Harmonic Variation* instead of Λ -variation and write H instead of Λ (HBV , $PHBV$, $HV(f)$, etc).

The notion of Λ -variation was introduced by Waterman [26] in one dimensional case and Sahakian [24] in two dimensional case. The notion of bounded partial variation (class PBV) was introduced by Goginava [10]. These classes of functions of generalized bounded variation play an important role in the theory of Fourier series.

We have proved in [14] the following theorem.

Theorem 1 (Goginava, Sahakian). *Let $\Lambda = \{\lambda_n = n\gamma_n\}$ and $\gamma_n \geq \gamma_{n+1} > 0$, $n = 1, 2, \dots$.*

1) *If*

$$(3) \quad \sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,$$

then $P\Lambda BV \subset HBV$.

2) *If, in addition, for some $\delta > 0$*

$$(4) \quad \gamma_n = O(\gamma_{n[1+\delta]}) \quad \text{as } n \rightarrow \infty$$

and

$$(5) \quad \sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

then $P\Lambda BV \not\subset HBV$.

Dyachenko and Waterman [8] introduced another class of functions of generalized bounded variation. Denoting by Γ the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2$ we define

$$\Lambda^*V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

Definition 4 (Dyachenko, Waterman). *Let f be a real function on I^2 . We say that $f \in \Lambda^*BV$ if*

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^*V(f) < \infty.$$

In [15] Goginava and Sahakian introduced a new classes of functions of generalized bounded variation and investigated the convergence of Fourier series of function of that classes.

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$ we denote

$$\Lambda^\# V_1(f) = \sup_{\{y_i\} \subset T} \sup_{\{I_i\} \in \Omega} \sum_i \frac{|f(I_i, y_i)|}{\lambda_i},$$

$$\Lambda^\# V_2(f) = \sup_{\{x_j\} \subset T} \sup_{\{J_j\} \in \Omega} \sum_j \frac{|f(x_j, J_j)|}{\lambda_j}.$$

Definition 5 (Goginava, Sahakian). *We say that the function f belongs to the class $\Lambda^\# BV$, if*

$$\Lambda^\# V(f) := \Lambda^\# V_1(f) + \Lambda^\# V_2(f) < \infty.$$

The following theorem is proved in [15]

Theorem 2. *a) If*

$$(6) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n \log(n+1)}{n} < \infty,$$

then

$$\Lambda^\# BV \subset HBV.$$

b) If $\frac{\lambda_n}{n} \downarrow 0$ and

$$\lim_{n \rightarrow \infty} \frac{\lambda_n \log(n+1)}{n} = +\infty,$$

then

$$\Lambda^\# BV \not\subset HBV.$$

In this paper we introduce new classes of bounded generalized variation.

Let f be a function defined on R^2 with 1- periodic relative to each variable. Δ_1 and Δ_2 is said to be a partitions with period 1, if

$$\Delta_i : \dots < t_{-1}^{(i)} < t_0^{(i)} < t_1^{(i)} < \dots < t_{m_i}^{(i)} < t_{m_i+1}^{(i)} < \dots, \quad i = 1, 2$$

satisfies $t_{k+m_i}^{(i)} = t_k^{(i)} + 1$ for $k = 0, \pm 1, \pm 2, \dots$, where $m_i, i = 1, 2$ are a positive integers.

Definition 6. *Let $p(n)$ be an increasing sequence such that $1 \leq p(n) \leq p$, $n \rightarrow \infty$, where $1 \leq p \leq +\infty$. We say that a function f belongs to the class $BV^\#(p(n) \uparrow p)$ if*

$$V_1^\#(f, p(n) \uparrow p) \equiv \sup_{\{y_i\} \subset I} \sup_{n \geq 1} \sup_{\Delta_1} \left\{ \left(\sum_{i=1}^{m_1} |f(I_i, y_i)|^{p(n)} \right)^{1/p(n)} : \inf_i |I_i| \geq \frac{1}{2^n} \right\} < +\infty,$$

and

$$V_2^\#(f, p(n) \uparrow p)$$

$$\equiv \sup_{\{x_j\} \subset I} \sup_{n \geq 1} \sup_{\Delta_2} \left\{ \left(\sum_{j=1}^{m_2} |f(x_j, J_j)|^{p(n)} \right)^{1/p(n)} : \inf_j |J_j| \geq \frac{1}{2^n} \right\} < +\infty,$$

where

$$I_i := (t_{i-1}^{(1)}, t_i^{(1)}), J_j := (t_{j-1}^{(2)}, t_j^{(2)}).$$

$C(I^2)$ and $B(I^2)$ are the spaces of continuous and bounded functions given on I^2 , respectively.

In this paper we prove inclusion relations between $\Lambda^\# BV$ and $BV^\#(p(n) \uparrow \infty)$ classes. In particular, the following are true

Theorem 3. $\Lambda^\# BV \subset BV^\#(p(n) \uparrow \infty)$ if and only if

$$(7) \quad \overline{\lim}_{n \rightarrow \infty} \sup_{1 \leq m \leq 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^m (1/\lambda_j)} < \infty.$$

Theorem 4. Let $\sum_{n=1}^{\infty} (1/\lambda_n) = +\infty$. Then there exists a functions $f \in BV^\#(p(n) \uparrow \infty) \cap C(I^2)$ such that $f \notin \Lambda^\# BV$.

Corollary 1. $BV^\#(p(n) \uparrow \infty) \subset \Lambda^\# BV$ if and only if $\Lambda^\# BV = B(I^2)$.

For the proof of this theorem the following lemma is needed:

Lemma 1. (see [16], p. 111) Let $0 \leq a_n \downarrow$, $0 \leq b_n \downarrow$, and let the relations

$$\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$$

be true for $k = 1, 2, \dots, m$. Then for convex functions Φ the inequality

$$\sum_{i=1}^m \Phi(a_i) \leq \sum_{i=1}^m \Phi(b_i)$$

holds.

Proof of Theorem 1. Let us take an arbitrary $f \in \Lambda^\# BV$. Without loss of generality it may be assumed that

$$|f(I_i, y_i)| \geq |f(I_{i+1}, y_{i+1})|, i = 1, \dots, m_1 - 1.$$

From the definition of $\Lambda^\#$ -bounded variation we have

$$\begin{aligned} \frac{|f(I_1, y_1)|}{\lambda_1} + \frac{|f(I_2, y_2)|}{\lambda_2} \dots + \frac{|f(I_{m_1}, y_{m_1})|}{\lambda_{m_1}} &\leq \Lambda^\# V_1(f) \\ \frac{|f(I_1, y_1)|}{\lambda_2} + \frac{|f(I_2, y_2)|}{\lambda_3} \dots + \frac{|f(I_{m_1}, y_{m_1})|}{\lambda_1} &\leq \Lambda^\# V_1(f) \\ &\vdots \end{aligned}$$

$$\frac{|f(I_1, y_1)|}{\lambda_{m_1}} + \frac{|f(I_2, y_2)|}{\lambda_1} + \dots + \frac{|f(I_{m_1}, y_{m_1})|}{\lambda_{m_1-1}} \leq \Lambda^\# V_1(f).$$

By summation we get

$$\sum_{i=1}^{m_1} \frac{1}{\lambda_i} \sum_{k=1}^{m_1} |f(I_k, y_k)| \leq \Lambda^\# V_1(f) m_1,$$

consequently,

$$\sum_{k=1}^{m_1} |f(I_k, y_k)| \leq \frac{\Lambda^\# V_1(f) m_1}{\sum_{i=1}^{m_1} (1/\lambda_i)} = \Lambda^\# V_1(f) \sum_{k=1}^{m_1} \frac{1}{\sum_{i=1}^{m_1} (1/\lambda_i)}.$$

If we take $a_k = |f(I_k, y_k)|$, $b_k = \frac{1}{\sum_{i=1}^{m_1} (1/\lambda_i)}$ and $\Phi(u) = u^{p(n)}$, and apply

Lemma 1 we get

$$\sum_{k=1}^{m_1} |f(I_k, y_k)|^{p(n)} \leq \Lambda^\# V_1(f)^{p(n)} \frac{m_1}{\left(\sum_{i=1}^{m_1} (1/\lambda_i)\right)^{p(n)}}.$$

Consequently, from the condition of the theorem we have

$$\begin{aligned} & \left(\sum_{k=1}^{m_1} |f(I_k, y_k)|^{p(n)} \right)^{1/p(n)} \leq \Lambda^\# V_1(f) \frac{m_1^{1/p(n)}}{\sum_{i=1}^{m_1} (1/\lambda_i)} \\ & \leq \Lambda^\# V_1(f) \sup_{1 \leq m \leq 2^n} \frac{m^{1/p(n)}}{\sum_{i=1}^m (1/\lambda_i)} < \infty. \end{aligned}$$

Analogously, we can prove that

$$\left(\sum_{k=1}^m |f(x_k, J_k)|^{p(n)} \right)^{1/p(n)} \leq \Lambda^\# V_2(f) \sup_{1 \leq m \leq 2^n} \frac{m^{1/p(n)}}{\sum_{i=1}^m (1/\lambda_i)} < \infty.$$

Therefore we proved that $f \in \Lambda^\# BV(p(n) \uparrow \infty)$.

Next, we suppose that the condition (7) does not satisfy. As an example we construct function from $\Lambda^\# BV$ which is not in $BV^\#(p(n) \uparrow \infty)$.

Since

$$\overline{\lim}_{n \rightarrow \infty} \sup_{1 \leq m \leq 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^m (1/\lambda_j)} = +\infty,$$

there exists a sequence of integers $\{n'_k : k \geq 1\}$ such that

$$(8) \quad \lim_{k \rightarrow \infty} \frac{m(n'_k)^{1/p(n'_k)}}{\sum_{j=1}^{m(n'_k)} (1/\lambda_j)} = +\infty,$$

where

$$\sup_{1 \leq m \leq 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^m (1/\lambda_j)} = \frac{m(n)^{1/p(n)}}{\sum_{j=1}^{m(n)} (1/\lambda_j)}.$$

We choose a monotone increasing sequence of positive integers $\{n_k : k \geq 1\} \subset \{n'_k : k \geq 1\}$ such that

$$(9) \quad \frac{m(n_k)^{1/p(n_k)}}{\sum_{j=1}^{m(n_k)} (1/\lambda_j)} \geq 4^k,$$

$$(10) \quad p(n_k) \geq n_{k-1},$$

$$(11) \quad n_k > 3n_{k-1} + 1 \quad \text{for all } k \geq 2.$$

From (8) and (10) it is evident that $2^{2n_{k-1}} < m(n_k) \leq 2^{n_k}$.

Two cases are possible:

a) Let there exists a monotone sequence of positive integers $\{s_k : k \geq 1\} \subset \{n_k : k \geq 1\}$ such that

$$(12) \quad 2^{2s_{k-1}} < m(s_k) \leq 2^{s_k - s_{k-1} - 1}.$$

Consider the function f_k defined by

$$f_k(x) = \begin{cases} h_k(2^{s_k}x - 2j + 1), & x \in [(2j-1)/2^{s_k}, 2j/2^{s_k}) \\ -h_k(2^{s_k}x - 2j - 1), & x \in [2j/2^{s_k}, (2j+1)/2^{s_k}) \\ & \text{for } j = m(s_{k-1}), \dots, m(s_k) - 1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$h_k = \left(\frac{1}{2^k \sum_{j=1}^{m(s_k)} (1/\lambda_j)} \right)^{1/2}.$$

Let

$$f(x, y) = \sum_{k=2}^{\infty} f_k(x) f_k(y),$$

where

$$f(x+l, y+s) = f(x, y), \quad l, s = 0, \pm 1, \pm 2, \dots$$

First we prove that $f \in \Lambda^\# BV$. For every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$ we get

$$\begin{aligned} \Lambda^\# V_1(f; p(n) \uparrow \infty) &\leq \sum_{j=1}^{\infty} \frac{|f(I_j, y_j)|}{\lambda_j} \\ &\leq 4 \sum_{i=1}^{\infty} h_i^2 \sum_{j=1}^{m(s_i)} \frac{1}{\lambda_j} = 4 \sum_{i=1}^{\infty} \frac{1}{2^i} = 4. \end{aligned}$$

Analogously, we can prove that

$$\Lambda^\# V_2(f; p(n) \uparrow \infty) \leq 4.$$

Next, we shall prove that $f \notin BV^\#(p(n) \uparrow \infty)$. By (11), (12) and from the construction of the function we get

$$\begin{aligned} &V_1(f; p(n) \uparrow \infty) \\ &\geq \left\{ \sum_{j=m(s_{k-1})}^{m(s_k)-1} \left| f\left(\frac{2j-1}{2^{s_k}}, \frac{2j}{2^{s_k}}\right) - f\left(\frac{2j}{2^{s_k}}, \frac{2j}{2^{s_k}}\right) \right|^{p(s_k)} \right\}^{1/p(s_k)} \\ &= \left\{ \sum_{j=m(s_{k-1})}^{m(s_k)-1} \left| \left(f_k\left(\frac{2j-1}{2^{s_k}}\right) - f_k\left(\frac{2j}{2^{s_k}}\right) \right) f_k\left(\frac{2j}{2^{s_k}}\right) \right|^{p(s_k)} \right\}^{1/p(s_k)} \\ &= h_k^2 (m(s_k) - m(s_{k-1}))^{1/p(s_k)} \\ &\geq c \frac{m(s_k)^{1/p(s_k)}}{2^k \sum_{j=1}^{m(s_k)} (1/\lambda_j)} \geq c 2^k \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Therefore we get $f \notin BV^\#(p(n) \uparrow \infty)$.

b) Let

$$2^{n_k - n_{k-1} - 1} < m(n_k) \leq 2^{n_k} \quad \text{for all } k > k_0.$$

Consider the function g_k defined by

$$g_k(x) = \begin{cases} d_k(2^{n_k}x - 2j + 1), & x \in [(2j-1)/2^{n_k}, 2j/2^{n_k}) \\ -d_k(2^{n_k}x - 2j - 1), & x \in [2j/2^{n_k}, (2j+1)/2^{n_k}) \\ 0, & \text{otherwise} \end{cases} \quad \text{for } j = 2^{n_{k-1} - n_{k-2}}, \dots, 2^{n_k - n_{k-1} - 1} - 1$$

where

$$d_k = \left(\frac{1}{2^k \sum_{j=1}^{m(n_k)} (1/\lambda_j)} \right)^{1/2}.$$

Let

$$g(x, y) = \sum_{k=k_0+2}^{\infty} g_k(x) g_k(y)$$

where

$$g(x+l, y+s) = g(x, y), \quad l, s = 0, \pm 1, \pm 2, \dots$$

For every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$ we get

$$\begin{aligned} & \sum_{j=1}^{\infty} \frac{|f(I_j, y_j)|}{\lambda_j} \\ & \leq 4 \sum_{i=k_0+1}^{\infty} d_i^2 \sum_{j=1}^{2^{n_i-n_{i-1}-1}} \frac{1}{\lambda_j} \\ & \leq 4 \sum_{i=k_0+1}^{\infty} d_i^2 \sum_{j=1}^{m(n_i)} \frac{1}{\lambda_j} < \infty. \end{aligned}$$

Analogously, we can prove that

$$\sum_{j=1}^{\infty} \frac{|f(x_j, J_j)|}{\lambda_j} < \infty.$$

Hence we have $g \in \Lambda^{\#}BV$.

Next we shall prove that $g \notin BV^\#(p(n) \uparrow \infty)$. By (8), (10), (11) and from the construction of the function we get

$$\begin{aligned}
& V_1^\#(g; p(n) \uparrow \infty) \\
& \geq \left\{ \sum_{j=2^{n_{k-1}-n_{k-2}}}^{2^{n_k-n_{k-1}-1}-1} \left| g\left(\frac{2j-1}{2^{n_k}}, \frac{2j}{2^{n_k}}\right) - g\left(\frac{2j}{2^{n_k}}, \frac{2j}{2^{n_k}}\right) \right|^{p(n_k)} \right\}^{1/p(n_k)} \\
& = \left\{ \sum_{j=2^{n_{k-1}-n_{k-2}}}^{2^{n_k-n_{k-1}-1}-1} \left| \left(g_k\left(\frac{2j-1}{2^{n_k}}\right) - g_k\left(\frac{2j}{2^{n_k}}\right) \right) g_k\left(\frac{2j}{2^{n_k}}\right) \right|^{p(n_k)} \right\}^{1/p(n_k)} \\
& = d_k^2 (2^{n_k-n_{k-1}-1} - 2^{n_{k-1}-n_{k-2}})^{1/p(n_k)} \\
& \geq \frac{1}{4} d_k^2 2^{(n_k-n_{k-1})/p(n_k)} \\
& \geq \frac{c 2^{n_k/p(n_k)}}{2^{k+2} \sum_{j=1}^{m(n_k)} (1/\lambda_j)} \\
& \geq c \frac{m(n_k)^{1/p(n_k)}}{2^k \sum_{j=1}^{m(n_k)} (1/\lambda_j)} \\
& \geq c 2^k \rightarrow \infty \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

Therefore we get $g \notin BV^\#(p(n) \uparrow \infty)$ and the proof of Theorem 1 is complete. \square

Proof of Theorem 2. We choose a monotone increasing sequence of positive integer $\{l_k : k \geq 1\}$ such that $l_1 = 1$ and

$$(13) \quad p(l_{k-1}) \geq \ln k \quad \text{for all } k \geq 2.$$

Set $(k = 1, 2, \dots)$

$$r_k(x) = \begin{cases} 2^{l_k+1} c_k (x - 1/2^{l_k}), & \text{if } 1/2^{l_k} \leq x \leq 3/2^{l_k+1} \\ -2^{l_k+1} c_k (x - 1/2^{l_k-1}), & \text{if } 3/2^{l_k+1} \leq x \leq 1/2^{l_k-1} \\ 0, & \text{otherwise} \end{cases}$$

where

$$c_k = \left(\sum_{j=1}^k \frac{1}{\lambda_j} \right)^{-1/4}$$

and

$$r(x, y) = \sum_{k=1}^{\infty} r_k(x) r_k(y)$$

where

$$r(x+l, y+s) = r(x, y) \quad l, s = 0, \pm 1, \pm 2, \dots$$

It is easy to show that function $r \in C(I^2)$.

First we show that $r \in BV^\#(p(n) \uparrow \infty)$. Let $\{I_i\}$ be an arbitrary partition of the interval I such that $\inf_i |I_i| \geq 1/2^l$. For this fixed l , we can choose integers l_{k-1} and l_k for which $l_{k-1} \leq l < l_k$ holds. Then it follows that $p(l_{k-1}) \leq p(l) \leq p(l_k)$ and $1/2^{l_k} < 1/2^l \leq 1/2^{l_{k-1}}$.

By (13) and from the construction of the function r we obtain

$$\begin{aligned} & \left\{ \sum_{i=1}^m |r(I_i, y_i)|^{p(l)} \right\}^{1/p(l)} \\ & \leq 4 \left\{ \sum_{i=1}^k c_i^{2p(l)} \right\}^{1/p(l)} \\ & \leq 2k^{1/p(l_{k-1})} \leq 2k^{1/\ln k} = 2e. \end{aligned}$$

Therefore $r \in BV^\#(p(n) \uparrow \infty)$ holds.

Finally, we prove that $r \notin \Lambda BV^\#$. Since $c_n \downarrow 0$, we get

$$\begin{aligned} & \sum_{j=1}^k \frac{|r(1/2^{l_j}, 3/2^{l_j+1}) - r(3/2^{l_j+1}, 3/2^{l_j+1})|}{\lambda_j} \\ & = \sum_{j=1}^k \frac{|(r_j(1/2^{l_j}) - r_j(3/2^{l_j+1})) r_j(3/2^{l_j+1})|}{\lambda_j} \\ & = \sum_{j=1}^k \frac{c_j^2}{\lambda_j} \geq c_k^2 \sum_{j=1}^k \frac{1}{\lambda_j} \\ & = \left(\sum_{j=1}^k \frac{1}{\lambda_j} \right)^{1/2} \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Therefore we get $r \notin \Lambda BV^\#$ and the proof of Theorem 2 is complete. \square

Since $\Lambda BV^\# = B(I^2)$ if and only if $\sum_{j=1}^{\infty} (1/\lambda_j) < \infty$ the validity of Corollary 1 follows from Theorem 2.

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